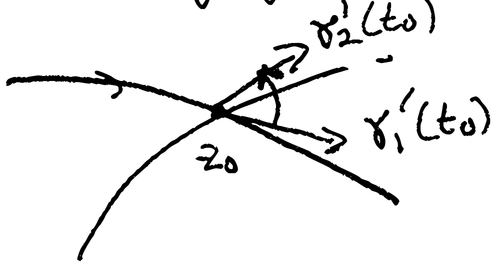
Lecture 13 (10/22/21)

Recall. • $\measuredangle(\gamma_1, \gamma_2)_{z_0}$ is signed angle between curves γ_1, γ_2 intersecting at z_0.



• Finish Lecture 12 notes from Thm 1.

Möbius Transformations, cont'd

• A Möbius $T(z) = \frac{az+b}{cz+d}$ has either ≤ 2 fixed on $\mathbb{C}_\infty$ or $T(z) = z$. To see this, solve for fixed points:

$$z = \frac{az+b}{cz+d} \Leftrightarrow cz^2 + dz = az + b$$

or

$$cz^2 + (d-a)z - b = 0. \qquad (*)$$

Two cases: 1) $c \neq 0$. Then $(*)$ has 2 sol'ns z_1, z_2, but $z_1 = z_2$ is a possibility (i.e. only 1 distinct sol'n).

Moreover, $T(\infty) = a/c \neq \infty$ so T has either 1 or 2 fixed points.

(ii) $c=0$. In this case, (*) becomes $(d-a)z - b = 0$, which has precisely one solution $z = \frac{b}{d-a}$ unless $d=a$ in which case either $\exists$ no sol'ns ($b \neq 0$) or all z are solutions. In latter case, $T = z$ and, in former, $T = z+b$. Note that in either case $T(\infty) = a/c = \infty$, so T has either 1 or 2 fixed points.

Most importantly, we obtain

<u>Prop 1.</u> If a Möbius T has 3 fixed points on $\mathbb{C}_\infty$, then $T(z) = z$.

Thus, you specify a Möbius uniquely once you specify its values at 3 points.

Can you specify T at 3 points z_1, z_2, z_3 arbitrarily? Yes, by the <u>cross ratio</u>: Let z_2, z_3, z_4 be distinct pts on $\mathbb{C}_\infty$. Then,

$$T(z) = \frac{z - z_3}{z - z_4} \cdot \left(\frac{z_2 - z_4}{z_2 - z_3} \right)$$

satisfies $T(z_2) = 1$, $T(z_3) = 0$, $T(z_4) = \infty$.

By Prop 1, it is unique T w/ this property. We call it the <u>cross ratio</u> and write $T(z) = (z, z_2, z_3, z_4)$.

Now, if you want to send z_2, z_3, z_4 to $w_2, w_3, w_4 \in \mathbb{C}_\infty$ let $T(z) = (z, z_2, z_3, z_4)$, $S(w) = (w, w_2, w_3, w_4)$ and set $R(z) = (S^{-1} \circ T)(z)$. R has desired property, and by Prop 1 it is unique.